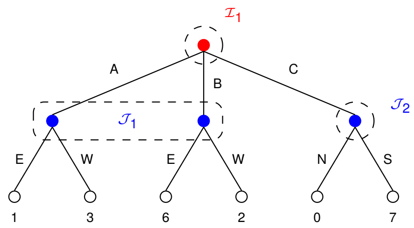
Problem 1. Behavioral strategies (4 points)

We aim to compute the behavioral saddle-point equilibrium of the zero-sum extensive form game shown below.

a) Verify that $\left(\begin{bmatrix}\frac{2}{3}\\\frac{1}{3}\end{bmatrix}, \begin{bmatrix}\frac{1}{6}\\\frac{5}{6}\end{bmatrix}\right)$ is a mixed-strategy equilibrium of $\begin{bmatrix}1 & 3\\6 & 2\end{bmatrix}$. What is the value of the game? (2 points)

b) Determine optimal strategy of player 2 (maximizer) at $\mathcal{J}_2$. (0.5 point)

c) Using the above two steps, determine the mixed strategy of player 1 at $\mathcal{I}_1$ and the strategy of player 2 for each of his information sets to characterize the behavioral saddle-point equilibrium of the game (1.5 points).



Problem 2. A zero-sum LQ game (7 points)

Consider a zero-sum linear quadratic game with two agents over two time steps. The state dynamics is

$$x_{t+1} = f(x_t, u_t, w_t) = \frac{1}{2}x_t + u_t + w_t, \ \forall t = 0, 1,$$

where $u_t \in \mathbb{R}$ and $w_t \in \mathbb{R}$ denote the control actions of the minimizer and the maximizer, respectively. The cost to optimize is $(x_2 - T)^2 + u_0^2 + u_1^2 - 2w_0^2 - 2w_1^2$. Our aim is to compute the feedback Nash equilibrium strategies over two time steps $t = 0, 1$, namely, $\sigma_t : \mathbb{R} \to \mathbb{R}$, for the minimizer, and $\gamma_t : \mathbb{R} \to \mathbb{R}$ for the maximizer.

a) We will use the dynamic programming approach. Given $V_2(x) = (x - T)^2$, complete the missing entries in the backward iteration for determining $V_1(x)$, namely, the cost-to-go at time 1. (1 point)

$$V_1(x) = \min_{u \in \mathbb{R}} \max_{?} \underbrace{\left[u^2 - 2w^2 + V_?\left(f(x, u, w)\right)\right]}_{J(x,u,w)} \tag{1}$$

b) Now, using the dynamics, write the expression for $J(x, u, w)$. (1 point)

c) Observe that $J(x, u, w)$ is convex and differentiable in $u \in \mathbb{R}$ and concave and differentiable in $w \in \mathbb{R}$. Thus, explain how you would determine the $\min_{u \in \mathbb{R}} J(x, u, w)$ and the $\max_{w \in \mathbb{R}} J(x, u, w)$. (.5 point)

d) Compute the feedback Nash equilibrium strategies at time $t = 1$ putting the steps above together. You should obtain a pair of affine strategies $\sigma_1(x) = k_u x + b_u$ and $\gamma_1(x) = k_w x + b_w$. (2 points) *You may use the fact that the inverse of a* 2×2 *matrix is given as:*

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix}d & -c\\-b & a\end{bmatrix}.$$

e) Using the strategies derived, compute the cost-to-go function $V_1(x)$. If you could not derive the explicit form of the strategies, you may substitute the affine strategies symbolically and continue. (1 point)

f) Now, write the backward iteration for computing $V_0(x)$. Would $\sigma_0(x), \gamma_0(x)$ be also affine? If so, would they have the same linear and offset terms (k_u, k_w, b_u, b_w)? (1.5 point)

Problem 3. Shortest path game (9 points + 1 bonus)

Alice and Bob play the following dynamic game: A token is moved along a directed graph $\mathcal{G}$ with nodes $\{s_1, s_2, \dots, s_6\}$ that represent the state of the game. At time $t = 0$, the token is placed at s_1. Edges are associated with costs, and we let $c_{ij} \in \mathbb{R}_{>0}$ denote the cost of the directed edge from s_i to s_j.

Alice and Bob have the same state-dependent action sets $\mathcal{U}_s = \mathcal{V}_s$. Namely, for each node s, we have $\mathcal{U}_s = \mathcal{V}_s = \mathcal{N}(s)$ where $\mathcal{N}(s)$ is the set of nodes v for which there is an edge from s to v. For odd i, the token transitions to the node determined by Alice's action $u \in \mathcal{U}_{s_i}$; then Alice incurs cost c_{ij}, and Bob incurs no cost. Conversely, for even i, the token transitions to the node determined by Bob's action $v \in \mathcal{V}_{s_i}$; then Bob incurs cost c_{ij}, and Alice incurs no cost. If the token is at s_6, it stays there and the cost is 0 for both players. The game ends at $t = 5$.

Figure 1 shows the graph $\mathcal{G}$. Both Alice and Bob are assumed to be cost minimizers.

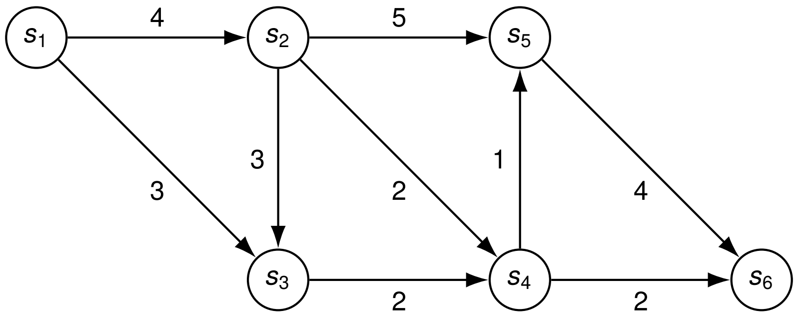


Figure 1

For $j \in \{1, 2, \dots, 6\}$ and $t \in \{0, 1, \dots, 5\}$, we define $V_t^A(s_j)$ as Alice's cost for the token to reach s_6 within at most t steps, assuming each player acts optimally with respect to her/his total cost). The respective cost incurred by Bob is $V_t^B(s_j)$. If s_6 is not reachable from s_j within at most t steps, we set the respective value to ∞.

a) Initializing $V_0^A(s) = 0$ for $s = s_6$ and $V_0^A(s) = \infty$ for $s \in \{s_1, \dots, s_5\}$, we can use dynamic programming to determine $V_t^A(s)$ for $t = 1, 2, \dots, 5$, and $s \in \{s_1, \dots, s_5\}$ as follows.

$$V_t^A(s_j) = \begin{cases} \min\limits_{s_k \in \mathcal{N}(s_j)} \left\{ c_{jk} + V_{t-1}^A(s_k) \right\}, & \text{if } j \text{ is odd;} \\ V_{t-1}^A(s_{k^\star}), & \text{otherwise, where } k^\star = \arg\min\limits_{s_k \in \mathcal{N}(s_j)} \left\{ c_{jk} + V_{t-1}^B(s_k) \right\}, \end{cases}$$

Write down the respective expression for determining $V_t^B(s_j)$. (1 point)

b) Let $u_t^\star(s)$ and $v_t^\star(s)$ be the action Alice and Bob must take in order to achieve cost $V_t^A(s)$ and $V_t^B(s)$, respectively. In Table 1 (next page), the box corresponding to state s and time t should contain $(V_t^A(s), V_t^B(s))$ at the top, and $(u_t^*(s), v_t^*(s))$ at the bottom (where "$-$" means this player can choose any action). Fill out the 6 missing boxes. (6 points)

c) Write down a strategy γ_A for Alice and γ_B for Bob such that (γ_A, γ_B) is a subgame perfect equilibrium of the above game. *Hint: Look at the last column of the table above.* (2 points)

d) (bonus) Now instead of having players minimize their own cost, suppose they aim to minimize the sum of both their total costs, i.e. the social cost. Write down a strategy $\widehat{\gamma}_A$ for Alice and $\widehat{\gamma}_B$ for Bob such that $(\widehat{\gamma}_A, \widehat{\gamma}_B)$ is a subgame perfect equilibrium of this modified game. *Hint: The strategies can be inferred directly by looking at the graph.* How does the social cost of $(\widehat{\gamma}_A, \widehat{\gamma}_B)$ compare to that of (γ_A, γ_B)? (1 point)

s \ t	0	1	2	3	4	5
s_1	(∞, ∞) $(-, -)$	(∞, ∞) $(-, -)$	(∞, ∞) $(-, -)$	$(4, 4)$ $(s_2, -)$	(,) (,)	(,) (,)
s_2	(∞, ∞) $(-, -)$	(∞, ∞) $(-, -)$	(,) (,)	(,) (,)	$(4, 3)$ $(-, s_4)$	$(4, 3)$ $(-, s_4)$
s_3	(∞, ∞) $(-, -)$	(∞, ∞) $(-, -)$	$(2, 2)$ $(s_4, -)$	$(6, 1)$ $(s_4, -)$	$(6, 1)$ $(s_4, -)$	$(6, 1)$ $(s_4, -)$
s_4	(∞, ∞) $(-, -)$	(,) (,)	(,) (,)	$(4, 1)$ $(-, s_5)$	$(4, 1)$ $(-, s_5)$	$(4, 1)$ $(-, s_5)$
s_5	(∞, ∞) $(-, -)$	$(4, 0)$ $(s_6, -)$	$(4, 0)$ $(s_6, -)$	$(4, 0)$ $(s_6, -)$	$(4, 0)$ $(s_6, -)$	$(4, 0)$ $(s_6, -)$
s_6	$(0, 0)$ $(-, -)$	$(0, 0)$ $(-, -)$	$(0, 0)$ $(-, -)$	$(0, 0)$ $(-, -)$	$(0, 0)$ $(-, -)$	$(0, 0)$ $(-, -)$

Table 1